§27. Scaling of the Distribution Function and the Critical Exponents near the Point of a Marginal Stability under the Vlasov-Poisson Equations

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Using direct integration of the Vlassov equation in configurational space, nonlinear dynamics of a model one-dimensional periodic self-gravitating system are investigated near the point of a marginal stability. The model of is described by the coupled Vlassov and Poisson equations under one-dimensional periodic geometry, i.e.

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial v} = 0, \qquad (1)$$

$$\frac{\partial^2 \Phi}{\partial x^2} = 4\pi G\left(\int_{-\infty}^{+\infty} f(x,v,t)\,dv - \langle\rho\rangle\right), \qquad (2)$$

$$f(0,v,t) = f(L,v,t), \qquad (3)$$

Here Φ is the gravitational potential, G is the gravitational constant, L is the length of the system. For the threshold phenomena, like the bump-on-tail instability studied by Berk et al. [1] the saturation is known due to wave-particle interaction in the separatrix area yields the scaling for the saturated amplitude of the electric field

$$E \sim \omega_L^2, \qquad (4)$$

where ω_L is the linear growth rate.

The nature of the saturation for the system described by equations (1)-(3) is the same: wave-particle interaction, but it is unaffected by additions like sources and sinks, introduced in [1].

The dispersion relation for the Maxwellian background near instability threshold is

$$\omega_m = i\sqrt{\frac{2}{\pi}}k_m\sigma\left[1 - \frac{\sigma^2}{\sigma_J^2(m)}\right], \qquad (5)$$

where $k_m = 2\pi m/L$ is the wavenumber, σ is the velocity dispersion,

$$\sigma_J^2(m) = \frac{4\pi G\rho_0}{k_m^2}, \qquad (6)$$

is the critical velocity dispersion for the wavenumber k_m. ρ_0 is the background density.

Near the instability threshold the following scaling has been calculated for the peak amplitude of the perturbation k_1

$$A \sim |\omega_1|^\beta, \qquad (7)$$

with $\beta = 1.907 \pm 0.006$.

At $\omega_1 = 0$ the response depends on the strain F_1 of external drive as

$$A \propto F_1^{1/\delta}, \qquad (8)$$

where $\delta = 1.544 \pm 0.002$.

The susceptibility

$$\chi = \frac{\partial A}{\partial F_1}, \quad F_1 \to 0, \qquad (9)$$

diverges as

$$\chi \propto |\omega_1|^{-\gamma_\pm} \qquad (10)$$

as $Im(\omega_1) \to \pm 0$, $\gamma_- = 1.020 \pm 0.008$ for $Im(\omega_1) < 0$, and $\gamma_+ = 0.995 \pm 0.020$ for $\omega_1 > 0$. Under this accuracy these critical exponents satisfy to the equality

$$\gamma_\pm = \beta(\delta - 1) \qquad (11)$$

known in the theory of critical phenomena as the Widom equality. Its existence is a direct consequence of scaling invariance of the distribution function at $|\theta| \ll 1$, i.e.

$$f_m(\lambda^{a_t}t,\ \lambda^{a_v}v,\ \lambda^{a_\theta}\theta,\ \lambda^{a_{A_0}}A_0,\ \lambda^{a_F}F_1) = \\ = \lambda f_m(t,v,\theta,A_0,F_1), \qquad (12)$$

[2]. Here f_m is a spatial Fourier component of the distribution function, i.e.

$$f(x,v,t) = \sum_{m=-\infty}^{\infty} f_m(v,t)\exp(ik_m x), \qquad (13)$$

$$\theta = \frac{\sigma^2 - \sigma_{cr}^2}{\sigma_{cr}^2}, \qquad (14)$$

is the dimensionless distance from the instability threshold for the wavenumber k_1, and

$$\sigma_{cr}^2 = \frac{4\pi G\rho_0}{k_1^2} \equiv \sigma_J^2(1) \qquad (15)$$

REFERENCES

[1] Berk, H.L., Breizman, B.N., Candy J., Pekker, M., Petviashvili, N.V. Phys. Plasmas **6**, (1999) 3102
[2] Ivanov, A.V., Astrophysical J. **550**, (2001) 622